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QUASILINEARIZATION AND THE SOLUTION OF NONLINEAR DESIGN PROBLEMS IN STRUCTURES UNDERGOING CREEP DEFORMATIONS

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Abstract—A class of multipoint-value problems involving generally nonlinear integro-differential equations of Volterra type, arising in applications of structural design in the presence of nonlinear creep is formulated and thoroughly investigated. First, a nonlinear multipoint-value problem is formulated and solved algorithmically by quasilinearization. This is done in Sections 2 and 3. Relaxation of design specifications yields a class of optimization problems whose solution is then outlined in Section 4. Sections 5–7 are devoted to computational aspects. In Section 6, a method to overcome some of the computational drawbacks of the classical Newton–Raphson–Kantorovich sequence in function space, is presented and applied to the problem under consideration. Section 7 deals with the reduction of large systems of Volterra integral equations to initial-value differential systems, while in Section 8 a numerical example is presented to illustrate the application and feasibility of the method.

1. INTRODUCTION

IN A recent paper [1] we have studied the solution of a nonlinear Volterra integral equation in the context of creep–buckling problems by means of a quasilinearization procedure. Here we extend the results of that analysis by considering a class of multipoint-value problems involving nonlinear integro-differential equations. Systems of this type arise naturally during the treatment of design and optimum-design problems in structures undergoing nonlinear creep deformations.

The equations of evolution of the system are given by (1) in the form of a generally nonlinear Volterra integral equation in terms of the state variable $\mathbf{u}(t)$, a vector of arbitrary dimension N. A general design problem is then formulated by adding a set of design specifications to the original dynamical system (1). First, a multipoint-value problem in terms of the augmented state vector $\mathbf{u}$ and design parameters $\mathbf{c}$, is stated and algorithmically solved by quasilinearization. This is done in Sections 2 and 3.

Relaxation of design specifications yields a class of optimization problems whose solution is outlined in Section 4. Sections 5–7 are devoted to computational aspects. In Section 6, a method to overcome some of the computational drawbacks of the classical Newton–Raphson–Kantorovich sequence in function space is presented and applied to the problem under consideration. Section 7 deals with the reduction of large systems of Volterra integral equations to initial-value differential systems, while in Section 8 a numerical example is presented to illustrate the application and feasibility of the method.

2. GENERAL DESIGN PROBLEM

Let $\mathbf{v}(t)$ and $\mathbf{u}(t)$ be two N-dimensional vectors representing a mechanical input–output pair of a given structure undergoing nonlinear creep deformations. In a number of cases the

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nonlinear functional relationship between $\mathbf{u}$ and $\mathbf{v}$ can be represented by the following nonlinear Volterra integral equation

$$\mathbf{v}(t) = \mathbf{h}(\mathbf{u}, \mathbf{c}) + \int_0^t \mathbf{F}(t-\tau)\mathbf{g}(\mathbf{u}, \mathbf{c})\,\mathrm{d}\tau, \tag{1}$$

where $\mathbf{F}$ is an $N \times N$ matrix, $\mathbf{h}$ and $\mathbf{g}$ are N-dimensional vectors and $\mathbf{c}$ is M-dimensional denoting the design vector. Typically, equation (1) arises when a conveniently discretized structure, in place of the original continuous one, is considered. Vector $\mathbf{h}$ accounts for the instantaneous response of the structural system, while the convolution term accounts for hereditary effects. Both vector functions $\mathbf{h}$ and $\mathbf{g}$ are assumed to be generally nonlinear, possessing all the continuity and convexity properties required in the subsequent analysis. The general nonlinear nature of $\mathbf{h}$ and $\mathbf{g}$ may be the consequence of both geometrical and material nonlinearities.

Generally, $\mathbf{v}$ is given while $\mathbf{u}$ must be computed together with the unknown design vector $\mathbf{c}$. Clearly, this problem is indeterminate unless appropriate additional information regarding the nature of the solution is provided. This is generally accomplished by adjoining a number of analytical design specifications of the form

$$k_j(\mathbf{u}(t_i)) = d_j, \qquad 0 \leq t_i \leq T, \qquad j = 1, 2, \ldots, M, \tag{2}$$

where T is the duration of the process. Now the design problem has a precise formulation in the form of the multipoint-value problem given by (1) and (2). Assuming that this problem has a solution and it is unique, we can compute $\mathbf{c}$ and $\mathbf{u}$ via a number of quasilinear procedures. The main purpose of this paper is to derive algorithmic solutions for this problem and to thoroughly discuss their numerical feasibility in the framework of digital computation.

3. QUASILINEARIZATION

Assuming the necessary convexity properties, $\mathbf{h}$ and $\mathbf{g}$ appearing in equation (1) can be represented in terms of the maximum operation, i.e.

$$\mathbf{h}(\mathbf{u}, \mathbf{c}) = \max_{\mathbf{x},\mathbf{y}}(\mathbf{h}(\mathbf{x}, \mathbf{y}) + \mathbf{H}_u(\mathbf{x}, \mathbf{y})(\mathbf{u}-\mathbf{x}) + \mathbf{H}_c(\mathbf{x}, \mathbf{y})(\mathbf{c}-\mathbf{y})), \tag{3}$$

$$\mathbf{g}(\mathbf{u}, \mathbf{c}) = \max_{\mathbf{x},\mathbf{y}}(\mathbf{g}(\mathbf{x}, \mathbf{y}) + \mathbf{G}_u(\mathbf{x}, \mathbf{y})(\mathbf{u}-\mathbf{x}) - \mathbf{G}_c(\mathbf{x}, \mathbf{y})(\mathbf{c}-\mathbf{y})), \tag{4}$$

where $\mathbf{H}_u$, $\mathbf{H}_c$, $\mathbf{G}_u$ and $\mathbf{G}_c$ are Jacobians given by

$$\begin{aligned}
\mathbf{H}_u &= \left(\frac{\partial h_i}{\partial u_j}\right), \qquad i, j = 1, 2, \ldots N,\\
\mathbf{H}_c &= \left(\frac{\partial h_i}{\partial c_j}\right), \qquad i = 1, 2, \ldots N \quad \text{and} \quad j = 1, 2, \ldots M,\\
\mathbf{G}_u &= \left(\frac{\partial g_i}{\partial u_j}\right), \qquad i, j = 1, 2, \ldots N,\\
\mathbf{G}_c &= \left(\frac{\partial g_i}{\partial c_j}\right), \qquad i = 1, 2, \ldots N \quad \text{and} \quad j = 1, 2, \ldots M,
\end{aligned} \tag{5}$$

where h_i, g_i, u_i and c_i are the components of vectors $\mathbf{h}$, $\mathbf{g}$, $\mathbf{u}$ and $\mathbf{c}$, respectively.

Introducing the integral operator L given by

$$L(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \mathbf{H}_u(\mathbf{x}, \mathbf{y})\mathbf{u} + \int_0^t \mathbf{F}(t-\tau)\mathbf{G}_u(\mathbf{x}, \mathbf{y})\mathbf{u}\, \mathrm{d}\tau, \tag{6}$$

linear in $\mathbf{u}$, substitution of (3) and (4) in (1) yields

$$L(\mathbf{u}, \mathbf{x}, \mathbf{y}) + \mathbf{K}(\mathbf{x}, \mathbf{y})\mathbf{c} \leq \mathbf{v} - \boldsymbol{\omega}(\mathbf{x}, \mathbf{y}), \tag{7}$$

where vector $\boldsymbol{\omega}$ and matrix $\mathbf{K}$ are given by

$$\begin{aligned} \boldsymbol{\omega}(\mathbf{x}, \mathbf{y}) = \mathbf{h}(\mathbf{x}, \mathbf{y}) - \mathbf{H}_u(\mathbf{x}, \mathbf{y})\mathbf{x} - \mathbf{H}_c(\mathbf{x}, \mathbf{y})\mathbf{y} + \int_0^t \mathbf{F}(t-\tau)\mathbf{g}(\mathbf{x}, \mathbf{y})\, \mathrm{d}\tau \\ - \int_0^t \mathbf{F}(t-\tau)\mathbf{G}_u(\mathbf{x}, \mathbf{y})\mathbf{x}\, \mathrm{d}\tau - \int_0^t \mathbf{F}(t-\tau)\mathbf{G}_c(\mathbf{x}, \mathbf{y})\mathbf{y}\, \mathrm{d}\tau \end{aligned} \tag{8}$$

and

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = \mathbf{H}_c(\mathbf{x}, \mathbf{y}) + \int_0^t \mathbf{F}(t-\tau)\mathbf{G}_c(\mathbf{x}, \mathbf{y})\, \mathrm{d}\tau, \tag{9}$$

respectively.

The integral inequality given by equation (7) can now be used to generate a variety of successive approximation schemes. For example, let $\mathbf{u}^{(n)}$ and $\mathbf{c}^{(n)}$ be the nth iterate of vectors $\mathbf{u}$ and $\mathbf{c}$, respectively. Then the system given by

$$L(\mathbf{u}^{(n+1)}, \mathbf{u}^{(n)}, \mathbf{c}^{(n)}) + \mathbf{K}(\mathbf{u}^{(n)}, \mathbf{c}^{(n)})\mathbf{c}^{(n+1)} = \mathbf{v} - \boldsymbol{\omega}(\mathbf{u}^{(n)}, \mathbf{c}^{(n)}) \tag{10}$$

where $\mathbf{u}^{(n+1)}$ satisfies the multipoint conditions

$$k_j(\mathbf{u}^{(n+1)}(t_i)) = d_j, \qquad j = 1, 2, \dots M, \tag{11}$$

and where $\mathbf{u}^{(0)}$ and $\mathbf{c}^{(0)}$ are arbitrary initial estimates, is immediately recognized to be the Newton–Raphson–Kantorovich scheme applied to our original system (1) and (2) [2]. It is of interest to note that these equations (10) could be directly derived by substituting $\mathbf{g}$ and $\mathbf{h}$ in equation (1) by the truncated functional expansions

$$\mathbf{h}(\mathbf{u}, \mathbf{c}) \cong \mathbf{h}(\mathbf{u}^{(n)}, \mathbf{c}^{(n)}) + \mathbf{H}_u(\mathbf{u}^{(n)}, \mathbf{c}^{(n)})(\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}) + \mathbf{H}_c(\mathbf{u}^{(n)}, \mathbf{c}^{(n)})(\mathbf{c}^{(n+1)} - \mathbf{c}^{(n)}) \tag{12}$$

and

$$\mathbf{g}(\mathbf{u}, \mathbf{c}) \cong \mathbf{g}(\mathbf{u}^{(n)}, \mathbf{c}^{(n)}) + \mathbf{G}_u(\mathbf{u}^{(n)}, \mathbf{c}^{(n)})(\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}) + \mathbf{G}_c(\mathbf{u}^{(n)}, \mathbf{c}^{(n)})(\mathbf{c}^{(n+1)} - \mathbf{c}^{(n)}), \tag{13}$$

respectively.

The sequences $\mathbf{u}^{(n)}$ and $\mathbf{c}^{(n)}$ given by equations (10) and (11) are known to be quadratically convergent when convergent at all. In general, convergence of the sequences will depend on the convexity of the operator in the neighborhood of the fixed point $\mathbf{u}$, $\mathbf{c}$ and the relative distance of the initial estimates $\mathbf{u}^{(0)}$ and $\mathbf{c}^{(0)}$ to $\mathbf{u}$ and $\mathbf{c}$, respectively. If, additionally, the integral operator enjoys some monotone property in the neighborhood of the solution, one can expect some monotonic behavior of the approximating sequences.

Our next step now is to derive an algorithmic solution for the problem given by equations (10) and (11). This is easily done in view of the linearity of the integral operator (10). In fact, let $\mathbf{U}^{(n+1)}$ be the $N \times M$ fundamental matrix satisfying the linear system

$$L(\mathbf{U}^{(n+1)}, \mathbf{u}^{(n)}, \mathbf{c}^{(n)}) = -\mathbf{K}(\mathbf{u}^{(n)}, \mathbf{c}^{(n)}), \tag{14}$$

and let $\mathbf{z}^{(n+1)}$ be the N-dimensional vector solution of

$$L(\mathbf{z}^{(n+1)}, \mathbf{u}^{(n)}, \mathbf{c}^{(n)}) = \mathbf{v} - \boldsymbol{\omega}(\mathbf{u}^{(n)}, \mathbf{c}^{(n)}). \tag{15}$$

Then $\mathbf{u}^{(n+1)}$ in equation (10) can be written

$$\mathbf{u}^{(n+1)} = \mathbf{z}^{(n+1)} + \mathbf{U}^{(n+1)}\mathbf{c}^{(n+1)}, \tag{16}$$

a representation of $\mathbf{u}^{(n+1)}$ linear in $\mathbf{c}^{(n+1)}$. Equations (11) provide the M conditions to determine vector $\mathbf{c}^{(n+1)}$ appearing in equation (16). If the functions k_j are linear in $\mathbf{u}(t_i)$, then substitution of (16) in (11) yields a linear system of equations in the components of vector $\mathbf{c}^{(n+1)}$. In general, a nonlinear system must be solved at each step of iteration. The process is continued until convergence is achieved.

Before considering a variety of interesting problems associated with the computational feasibility of equations (14) and (15), we shall be concerned with some questions derived from specific design considerations.

4. RELAXATION OF DESIGN SPECIFICATIONS

In general, the set of design specifications given by equations (2) are either very difficult or impossible to satisfy exactly. In fact, conditions for the existence of a solution of the multipoint value problem given by (1) and (2) might be too restrictive to be of interest on engineering grounds; therefore, the interest in reformulating the problem in a slightly modified manner such as to account for more realistic design considerations. This can be done in several ways. Here we only indicate one possible path of action, namely, the substitution of an exact multipoint-value problem by an approximated one via the application of some optimization procedure. For example, if we denote with $\mathbf{k}$ and $\mathbf{d}$ the vectors whose components are k_j and d_j, respectively, instead of equation (2) we wish to consider the less stringent conditions

$$\min_{c} \|\mathbf{k}(\mathbf{u}) - \mathbf{d}\| \tag{17}$$

where $\|\mathbf{x}\|$ denotes the norm of $\mathbf{x}$ defined in a suitable manner. As usual, candidates for norms are, among others, the least squares

$$\int_0^T \overline{(\mathbf{k}(\mathbf{u}) - \mathbf{d})}\Gamma(\mathbf{k}(\mathbf{u}) - \mathbf{d})\, d\tau \tag{18}$$

where the upper bar indicates transpose and Γ is a matrix, or the Chebyshev norm

$$\max_{0 \le t \le T} |\mathbf{k}(\mathbf{u}) - \mathbf{d}|. \tag{19}$$

It is clear that the same procedure derived in previous sections is applicable to the problem formulated here. For example, if $\mathbf{k}$ is linear in $\mathbf{u}$, substitution of (16) in (18) yields a linear least square problem while substitution of (16) in (19) and further minimization yields a minimax problem that can be solved by standard linear programming procedures. In general, a number of techniques are available to treat the nonlinear case.

Optimal design

The success of quasilinearization as applied in this work relies on the ability to exhibit an explicit dependence of $\mathbf{u}$ upon the design vector $\mathbf{c}$ at each step of iteration. This permits the formulation of an algebraic problem for **c** by using the given design specifications such as in equation (2) or the objective functions in equations (18) and (19). It is then clear that along the same lines a variety of problems arising in the theory of optimum structural design can be solved. For example, integral side conditions such as those arising from limitation of volume of the structure can be handled by Lagrange multipliers or Courant parameters, thus reducing the problem of optimal design to one similar to that treated in this paper. Problems of this kind will be treated separately.

5. COMPUTATIONAL ASPECTS

We have shown how a quasilinearization scheme can be used to effectively solve a variety of design and optimum design problems. It remains now to discuss computational feasibility in connection with the use of a digital computer. In fact, several difficulties may arise in connection with the numerical solution of the equations. In the first place, a straightforward numerical solution of the fundamental system given by equations (14) and (15) would generally require storage capacity for the present and previous iteration of vector $\mathbf{u}(t)$ over the whole interval of interest $0 \leq t \leq T$. This is due to the fact that Newton's method requires knowledge of the previous approximation, while the solution of systems of linear Volterra equations would require the storage of the present approximation. If the dimensions of N and M are relatively large, storage can be a severe limitation. Thus, the need for special consideration of feasible numerical schemes. This is presented in the following two sections.

6. AN ALTERNATIVE QUASILINEARIZATION SCHEME

The fundamental system given by equations (14) and (15) can be regarded as a system of $N(M+1)$ linear integral Volterra equations in $\mathbf{u}^{(n+1)}$, whose forcing terms and kernels depend on the previous approximation $\mathbf{u}^{(n)}$. This implies knowledge of vector $\mathbf{u}^{(n)}$ during the computation of the $n+1$ approximation. A different way of looking at the same problem is to consider the augmented $(n+1)N(M+1)$ system given by

$$L(\mathbf{U}^{(i+1)}, \mathbf{u}^{(i)}, \mathbf{c}^{(i)}) = -\mathbf{K}(\mathbf{u}^{(i)}, \mathbf{c}^{(i)}) \tag{20}$$

and

$$L(\mathbf{z}^{(i+1)}, \mathbf{u}^{(i)}, \mathbf{c}^{(i)}) = \mathbf{v} - \boldsymbol{\omega}(\mathbf{u}^{(i)}, \mathbf{c}^{(i)}), \qquad i = 0, 1, \ldots n+1, \tag{21}$$

where $\mathbf{c}^{(0)}$ is an initial estimate of the design vector and $\mathbf{u}^{(0)}$ is an initial estimate for $\mathbf{u}$, which can be computed from the equation

$$\mathbf{v} = \mathbf{h}(\mathbf{u}^{(0)}, \mathbf{c}^{(0)}) + \int_0^t \mathbf{F}(t-\tau)\mathbf{g}(\mathbf{u}^{(0)}, \mathbf{c}^{(0)})\,\mathrm{d}\tau, \tag{22}$$

or by any other procedure. Clearly, this method avoids the necessity of storing the previous approximation, because at any stage $n+1$, not only $\mathbf{u}^{(n+1)}$ but also $\mathbf{u}^{(n)}$, $\mathbf{u}^{(n-1)}, \ldots, \mathbf{u}^{(1)}$ are being computed simultaneously.

At the same time that this method reduces storage, especially if the integral equations are reduced to differential equations, it requires the solution of a larger number of equations, namely,

$$\tfrac{1}{2}n(n+1)N(M+1)$$

for n iterations, in contrast with the $N(M+1)$ equations required when the previous approximation is available. This is not a severe limitation if n is small, but in any case it is convenient to seek alternative methods which do not present those restrictions. To overcome most of the difficulties of time and storage previously discussed in connection with the Newton–Raphson–Kantorovich approach in function space, equation (7) can be used as the starting point to implement an alternative quasilinear procedure. The basic idea of this method is the following: instead of using the last approximation in place of $\mathbf{x}$ in equation (7), we compute a function $\mathbf{x}(\mathbf{y})$ as the solution of the original functional equation (1), where the design vector $\mathbf{c}$ is replaced by the last approximation obtained for $\mathbf{y}$. In this fashion we are led to solve at each step of iteration, the following system

$$\begin{aligned} L(\mathbf{U}^{(n+1)}, \mathbf{x}, \mathbf{c}^{(n)}) &= -\mathbf{K}(\mathbf{x}, \mathbf{c}^{(n)}), \\ L(\mathbf{z}^{(n+1)}, \mathbf{x}, \mathbf{c}^{(n)}) &= \mathbf{v} - \boldsymbol{\omega}(\mathbf{x}, \mathbf{c}^{(n)}), \\ \mathbf{h}(\mathbf{x}, \mathbf{c}^{(n)}) + \int_0^t \mathbf{F}(t-\tau)\mathbf{g}(\mathbf{x}, \mathbf{c}^{(n)})\,\mathrm{d}\tau &= \mathbf{v}. \end{aligned} \tag{23}$$

Clearly, substitution of $\mathbf{U}^{(n+1)}$ and $\mathbf{z}^{(n+1)}$ given by equations (23) in (16) yields the desired new approximation for $\mathbf{u}$. This approach is particularly powerful if it is used in conjunction with the reduction of equations (23) to a system of ordinary differential equations, a technique to be treated in the next section.

The sequence generated by this method is indeed quadratically convergent when convergent at all. Abundant numerical experimentation has shown to the author of this work the merits of the present approach. However, it was found in a number of cases that the success of this method relies on the availability of a better estimate for the design vector $\mathbf{c}$ than is required when the Newton–Raphson–Kantorovich scheme is used.

7. REDUCTION TO INITIAL-VALUE DIFFERENTIAL SYSTEMS

Straightforward quadrature techniques for the solution of the integral-initial-value problem given by the system of integral equations (14), (15) or (20), (21) or (23) require storage of the function currently computed over the whole interval, $0 \leq t \leq T$, [3]: thus, the interest in transforming this problem into a differential-initial-value problem. This can be done in a number of ways. As an illustration, we consider a scalar nonlinear integral equation of the type

$$u(t) + \int_0^t f(t-\tau)g(u)\,\mathrm{d}\tau = v(t). \tag{24}$$

If the kernel $f(t)$ is given as an expansion of exponential functions of the form

$$f(t) = \sum_{i=1}^{R} c_i\,\mathrm{e}^{-r_i t}, \tag{25}$$

then the reduction to a differential form is immediate. In fact, substitution of (25) in (24) yields

$$u(t) = v(t) - \sum_{i=1}^{R} c_i z_i(t), \tag{26}$$

where the functions

$$z_i(t) = \mathrm{e}^{-r_i t} \int_0^t g(u(\tau))\, \mathrm{e}^{r_i \tau}\, \mathrm{d}\tau, \qquad i = 1, 2, \ldots R, \tag{27}$$

clearly satisfy the differential equations

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} + r_i z_i = g(u), \tag{28}$$

or, on account of equation (26),

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} + r_i z_i = g\left(v - \sum_{1}^{R} c_j z_j\right), \qquad i = 1, 2, \ldots R \tag{29}$$

a coupled, nonlinear system of differential equations of order R subject to the initial conditions

$$z_i(0) = 0. \tag{30}$$

When, instead of a scalar integral equation such as (24), we consider a system of $N \times N$ of such equations, the number of differential equations which will result is $R \times N^2$. The number of ordinary differential equations that can be integrated in a routine way on a digital computer is of the order of several thousands. This gives an idea of the size of the problem that can be treated by this method.

A severe limitation of the method just described is the requirement that f in equation (24) be given in terms of exponentials. When the kernel is not originally available in the form given by equation (25), an approximated expansion must be obtained. This approximation is unfortunately not a routine matter [4]. A method to circumvent the difficulties associated with the approximation of functions with exponentials was presented in [5] and [6] in connection with the identification problem in viscoelasticity.

8. EXAMPLE

In [1] it has been shown that the deflections $u(t)$ in the middle of an axially-loaded H column undergoing nonlinear creep deformations whose $\sigma - \varepsilon$ law is given by

$$\varepsilon = \sigma/E + \int_0^t (\sigma/E^*)^3 f(t-\tau)\, \mathrm{d}\tau, \tag{31}$$

where f, a kernel associated with the primary creep of the material, is given by equation (25), E is the instantaneous modulus of elasticity and E^* a material constant defining the rate of creep in the secondary stage, satisfies the following nonlinear integral Volterra equation

$$u(t) - \lambda \int_0^t (3u + u^3) f(t-\tau)\, \mathrm{d}\tau = \gamma \tag{32}$$

where λ and γ are coefficients involving a number of geometrical and mechanical parameters whose specification is not essential in this example. Clearly, equation (32) uniquely determines $u(t)$ in the semi-open interval $0 \leq t < t_{cr}$, where t_{cr} is the critical time, when λ and γ are given. Our problem here is to compute the value of λ for which the deflection u reaches a prescribed value d at a given time $t_1 < t_{cr}$. Following the procedure given at the end of Section 6, we consider the linearized equation for $u^{(n+1)}$

$$\begin{aligned} u^{(n+1)} - 3\lambda^{(n)} \int_0^t (1+v^2)u^{(n+1)} f(t-\tau)\,\mathrm{d}\tau \\ = \gamma + 3\lambda^{(n)} \int_0^t (1+v^2)vf(t-\tau)\,\mathrm{d}\tau - \lambda^{(n+1)} \int_0^t (3v+v^3) f(t-\tau)\,\mathrm{d}t \end{aligned} \tag{33}$$

where v is the solution of

$$v - \lambda^{(n)} \int_0^t (3v+v^3) f(t-\tau)\,\mathrm{d}\tau = \gamma. \tag{34}$$

The auxiliary condition is the design specification

$$u^{(n+1)}(t_1) = d. \tag{35}$$

We consider now functions $u_1^{(n+1)}$ and $u_2^{(n+1)}$ satisfying

$$u_1^{(n+1)} - 3\lambda^{(n)} \int_0^t (1+v^2)u_1^{(n+1)} f(t-\tau)\,\mathrm{d}\tau = \gamma + 3\lambda^{(n)} \int_0^t (1+v^2)vf(t-\tau)\,\mathrm{d}\tau \tag{36}$$

and

$$u_2^{(n+1)} - 3\lambda^{(n)} \int_0^t (1+v^2)u_2^{(n+1)} f(t-\tau)\,\mathrm{d}\tau = -\int_0^t (3v-v^3) f(t-\tau)\,\mathrm{d}\tau, \tag{37}$$

respectively. Clearly we have

$$u^{(n+1)} = u_1^{(n+1)} + \lambda^{(n+1)} u_2^{(n+1)}. \tag{38}$$

Using equations (35) and (38), we can immediately compute the value of the design parameter at the $n+1$ iteration by means of

$$\lambda^{(n+1)} = \frac{d - u_1^{(n+1)}(t_1)}{u_2^{(n+1)}(t_1)} \tag{39}$$

for which it is necessary to compute the values of $u_1^{(n+1)}$ and $u_2^{(n+1)}$ at $t = t_1$. To do so, we first reduce equations (34), (36) and (37) to a system of ordinary differential equations. Remembering that f is given by equation (25), we introduce the following functions

$$x_i = \mathrm{e}^{-r_i t} \int_0^t (1+v^2) u_1^{(n+1)}\, \mathrm{e}^{r_i \tau}\,\mathrm{d}\tau, \tag{40}$$

$$y_i = \mathrm{e}^{-r_i t} \int_0^t (1+v^2) u_2^{(n+1)}\, \mathrm{e}^{r_i \tau}\,\mathrm{d}\tau, \tag{41}$$

$$z_i = \mathrm{e}^{-r_i t} \int_0^t (3v+v^3)\, \mathrm{e}^{r_i \tau}\,\mathrm{d}\tau, \tag{42}$$

$$\omega_i = \mathrm{e}^{-r_i t} \int_0^t (1+v^2)v\, \mathrm{e}^{r_i \tau}\,\mathrm{d}\tau, \qquad i = 1, 2, \ldots R, \tag{43}$$

where to simplify the notation the superscript $n+1$ indicating the number of the iteration has not been written. Clearly, these four functions satisfy the following system of differential equations

$$\frac{dx_i}{dt}+r_ix_i = (1+v^2)u_1^{(n+1)}, \qquad x_i(0) = 0, \tag{44}$$

$$\frac{dy_i}{dt}+r_iy_i = (1+v^2)u_2^{(n+1)}, \qquad y_i(0) = 0, \tag{45}$$

$$\frac{dz_i}{dt}+r_iz_i = (3+v^2)v, \qquad z_i(0) = 0, \tag{46}$$

$$\frac{d\omega_i}{dt}+r_i\omega_i = (1+v^2)v, \qquad \omega_i(0) = 0, \tag{47}$$

$i = 1, 2, \ldots R$. Functions v, $u_1^{(n+1)}$ and $u_2^{(n+1)}$ in equations (34), (36) and (37) can be written in terms of x_i, y_i, z_i and ω_i, i.e.

$$v = \gamma+\lambda^{(n)}\sum_{j=1}^{R} c_jz_j, \tag{48}$$

$$u_1^{(n+1)} = \gamma+3\lambda^{(n)}\sum_{j=1}^{R} c_j(x_j+z_j), \tag{49}$$

$$u_2^{(n+1)} = \sum_{j=1}^{R} c_j(3\lambda^{(n)}y_j-\omega_j). \tag{50}$$

Equations (48)–(50) follow from the substitution of (40)–(43) in (34), (36) and (37). Now, integration of the differential initial-value (44)–(47) together with the auxiliary equations (48)–(50) yields the two fundamental solutions $u_1^{(n+1)}$ and $u_2^{(n+1)}$ from which $\lambda^{(n+1)}$ can be computed using equation (39).

This procedure is now illustrated numerically by the following example. We consider the kernel $f(t)$ given by

$$f(t) = 1+0{\cdot}3\,e^{-t}+0{\cdot}2\,e^{-3t} \tag{51}$$

and $\gamma = 1$, $\lambda = 0{\cdot}100$. Then equation (32), conveniently reduced to a system of ordinary differential equations subject to initial values, yields a deflection $u(t)$. The critical time for this problem was found to be approximately $t_{cr} = 1{\cdot}78$. The deflection u at $t = 1$ was found to be

$$u(1) = 1{\cdot}90490. \tag{52}$$

Using (52) as a design specification, we integrate the system given by equations (44)–(47) together with the auxiliary equations (39), (48)–(50). The initial approximation for λ was taken to be $\lambda^{(0)} = 0{\cdot}15$. The integration was performed with an Adams–Moulton scheme using 0·01 as the step of integration over the interval (0, 1). Six iterations were necessary to repeat five significant places in the deflection $u(t)$ and the design parameter λ. In general, the number of iterations for a given accuracy will depend on the initial approximation used for λ. The values of the design parameter λ at each step of the iterative process is given in the table below:

TABLE 1

n	$\lambda^{(n)}$
Initial estimate	0·1500
1	0·1338
2	0·1154
3	0·1036
4	0·1004
5	0·1000

The associated deflection histories are presented in Fig. 1.

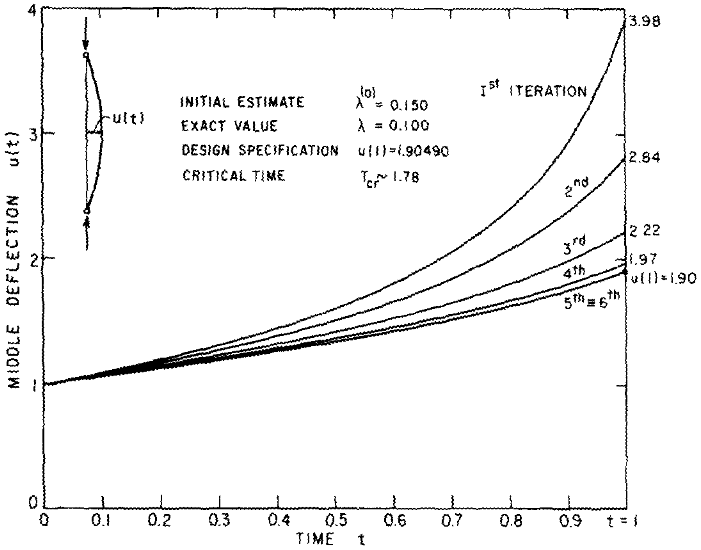

FIG. 1. Deflection histories associated with each iteration.

9. CONCLUSIONS

The main purpose of this study has been to present a thorough treatment of the numerical solution of a class of nonlinear Volterra integro-differential systems associated with problems of structural design in the presence of nonlinear creep.

The study comprises the mathematical formulation of a family of structural design problems and its numerical treatment by using ideas of quasilinearization and differential approximation. An efficient quasilinear procedure, that eliminates the need to store the previous numerical approximation, has been introduced in combination with the reduction of Volterra integral equations to differential equations subject to initial values, in an effort to overcome problems of storage typically associated with systems of Volterra equations.

The example of Section 8 has been presented to familiarize the reader with the application of the method. The accuracy of the numerical procedures in the present example is stressed, particularly in view of the fact that the design parameter λ is an unstable functional of the deflection history $u(t)$, $0 \leq t \leq t_1$, as t_1 approaches the critical time t_{cr} thus making the determination of λ a highly sensitive process.

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REFERENCES

[1] N. Distéfano and J. Sackman, Quasilinearization and the Computation of the Deflections of Metal Structures Undergoing Creep, ***Proceedings Second IUTAM Symposium on Creep in Structures***, Gotheburg, Sweden (1970). See also SESM Report 69-27, University of California, Berkeley.

[2] R. Bellman and R. Kalaba, ***Quasilinearization and Nonlinear Boundary-Value Problems.*** Elsevier (1965).

[3] J. C. Jones, On the numerical solution of convolution integral equations and systems of such equations. *Maths Comput.* **15**, 131 (1961).

[4] C. Lanzos, *Applied Analysis.* Prentice-Hall (1956).

[5] N. Distéfano, On alternative representations of time varying viscoelastic materials. *Int. J. Solids Struct.* **6**, 1021 (1970).

[6] N. Distéfano, On the identification problem in linear viscoelasticity. *Z. angew. Math. Mech.* **50**, 683–690 (1970).

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Абстракт—Предлагается и тщательно исследуется класс задач многоточенымх значений, вызывающих вообще нелинейные интегрально-дифференциальные уравнения типа Вольтерры, являющимися результатом применений, при расчетах конструкций, учета нелинейной ползучести. Сначала, формулируется и решается алгорифмически, путём квазилинеаризации, нелинейная задача многоточенного значения. Зто предложено в частях 2 и 3. Релаксация подробностей проектирования даёт класс оптимализации, решение когорого подчеркнуто в части 4. Части 5, 6 и 7 посвящены расчетным аспектам. В части 6, даётся и применятся к задаче, при некоторых условиях, метод для учета некоторых недостатков классического исследование Ньютона-Рафсона-Канторовича в функциональном пространстве. Часть 7 касается сведения больших систем интегральных уравнений Вольтерры к дифференциальным системам с начальными значениями. В части 8 представляется числовый пример для иллюстрации применимости метода.